Noise strength effects on the relaxation properties of weakly coupled Ginzburg-Landau models

Bruno Mota[1] and Emmanuel Pereira[1,2,*]
[1]*Departmento Física-ICEx, UFMG, Caîxa Postal 702, 30.161-970 Belo Horizonte MG, Brazil*
[2]*Center for Mathematical Science Research, Rutgers University, 110 Frelinghuysen Road, Piscataway, New Jersey 08854-8019*
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We search for new effects and extend previous investigations on the relaxation properties of weakly coupled stochastic Ginzburg-Landau models under noise with different intensity. We calculate the one-particle mass up to second order in the Ginzburg-Landau potential coupling constant and show that, as previously predicted, it is in fact less sensitive to the noise strength than the two-particle bound state. For $d=1$ and 2 and a negative quartic term in the potential, we show that the difference between these masses becomes smaller as we increase the noise strength, which indicates the possibility of a crossover for large noise (i.e., the two-particle bound state mass may become smaller than the one-particle mass) or of a phase transition (the masses are going to zero). For $d=1$ and in the ladder approximation (first order in the coupling constant), we show the absence of resonances in the spectrum close to the two-particle threshold and the existence of one antibound state for a positive quartic term in the potential, which indicates the possibility of some bound state due to changes in the system (e.g., in the noise intensity).

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Noise and nonequilibrium processes are everywhere. Moreover, the noise effects are far from trivial: their influence on dynamical properties of many different systems in physics, biology, mathematics, and other fields are remarkable (e.g., stochastic resonance, noise-induced phase transition, etc.) and sometimes even counterintuitive, which makes the study of noise effects on the basic properties of general stochastic models a subject of fundamental relevance.

In this paper we consider the (extensively used) Ginzburg-Landau (GL) model with a Langevin dynamics, and extend the analysis of a previous work of one of the authors [1] to learn more about the spectrum of the generator of the dynamics. In particular, we search for new effects and investigate in more detail the action of the noise strength on the relaxation properties of such systems.

Let us present the model. We consider scalar systems in a space lattice (to avoid the ultraviolet problem, which is irrelevant for the low-lying spectrum) with stochastic dynamics given by the Langevin equation

$$(\partial/\partial t)\,\varphi(\vec{x},t)=-\tfrac{1}{2}\nabla S(\varphi(\vec{x},t))+\eta(\vec{x},t), \tag{1}$$

where $\varphi(\vec{x},t)\in\mathbb{R}$, $t\in[0,\infty)$, $\nabla S=\delta S/\delta\varphi$ with

$$S(\varphi)=\sum_{\vec{x}\in\mathbb{Z}^d}\left\{\frac{1}{2}\varphi(\vec{x})[(-\Delta+m^2)\varphi](\vec{x})+\lambda\mathcal{P}(\varphi(\vec{x}))\right\}, \tag{2}$$

where Δ is the lattice Laplacian, λ and m are positive parameters ($\lambda\ll 1\ll m$), and $\mathcal{P}$ is an even polynomial. η is a Gaussian white-noise random variable with the expectations

$$E(\eta(\vec{x},t))=0,\quad E(\eta(\vec{x},t)\,\eta(\vec{y},t'))=\gamma\delta_{\vec{x},\vec{y}}\delta(t-t'),$$

γ is positive (the noise strength). These models (among many other applications [2,3]) are recurrent in the study of dynamical critical phenomena (e.g., magnetic systems). In Ref. [4] a simple version of it is used to describe a system without disorder or explicit frustation with stretched-exponential relaxation (this regime, of course, depends on the choice of the involved parameters).

In Ref. [1], the effects of changes in the noise intensity in the low-lying spectrum of the stochastic Langevin dynamics generator associated to these massive weakly coupled GL models are analyzed. The stochastic problem is first mapped into a quantum field theory via a Feynman-Kac formalism (standard in quantum physics and field theory [5]). Then, the Bethe-Salpeter (BS) equation for the field correlation functions is studied in the ladder approximation (up to first order in the coupling constant λ), in order to search for the presence of bound states of two particles. For the space dimension $d=1$ and 2, the presence of one bound state is shown for a negative quartic term in the GL potential, even for small noise. In fact, for $\gamma=1$ this result is rigorously proved in [6] (besides the existence of an isolated dispersion curve for the one-particle massive spectrum; other rigorous results have also been established in agreement with some previous analysis using a ladder approximation in Refs. [7,8]). For large noise (precisely, $\lambda\gamma\approx m^{\alpha}$, where $\alpha=2,4,6$ or 8), it is shown, also in Ref. [1], that a two-particle bound state appears and disappears again (increasing the noise intensity) also for $d=3$ and a negative quartic term in the GL potential, as well as for $d\leqslant 3$ and a positive quartic term. Some curves for the bound state mass are depicted below, showing (in the ladder approximation) that a bound state mass appears and its value decays as we increase the noise strength. Comparing to the one-particle mass computated in first order in λ, the curves predict (for some parameters) a very interesting phenomenon: the crossover between the masses of the one-particle and the two-particle bound state for a suitable noise strength: i.e., the bound state mass may become smaller than the one-particle mass. We emphasize that these spectral results show up directly in the approach to equilibrium: e.g., for a magnetic system ruled by a stochastic GL model (in this case, φ is the magnetization), the relaxation (in time) of the

*Author to whom correspondence should be addressed. Email address: epereira@math.rutgers.edu

magnetization fluctuation is given by the two-point correlation $\langle \varphi(t)\varphi(0)\rangle$, which behaves like as $\exp[-Mt]$, where M is the one-particle mass, and the fluctuation of the susceptibility is given by $\langle \varphi(t)\varphi(t);\varphi(0)\varphi(0)\rangle$ (where ; stands for the truncated correlation) which goes as $\exp[-M^*t]$, and M^* is the two-particle bound state mass. We also remark that these results follow for another similar dynamical system: for imaginary time, the Langevin equation becomes a nonlinear Schrödinger equation with the spectrum keeping the properties described here.

However, the predicted mass crossover phenomenon and the bound state for the potential with positive quartic term described above involve noise with intensity in a range that makes the question of the validity of the ladder approximation. In this paper, we perform further analysis on the spectrum of the generator of the dynamics, which, besides showing us new spectral properties, sheds some light on the previous question. We study the two-point function up to second order in λ (we remark that the term order λ in the one-particle mass vanishes, which makes unclear the dependence of such mass on the noise strength considering only the ladder approximation). We show that the one-particle mass also decreases with the noise but the bound state mass decreases much rapidly (it happens still in the region where the perturbative approach is certainly correct), a clear evidence in favor of the crossover phenomenon. For $d=1$, in the ladder approximation and in the region close to the two-particle threshold, we show the presence of an antibound state for a positive quartic term in the GL potential, and also the absence of resonance (for positive and negative terms). We remind that the existence of an antibound state close to the two-particle threshold indicates the possibility of appearance of a bound state due to small changes in the interaction. In short, the presence of this antibound state in the perturbative regime may be an indication of a bound state for large noise.

Now we turn to the technical description of our problem. For any function $f(\varphi)$ with evolution given by $f_t(\psi)=E[f(\varphi(t))]$, where $\varphi(0)=\psi$ is some initial condition in Eq. (1), the Langevin dynamics described by Eq. (1) leads to the time dependence $f_t(\psi)=e^{-tH}f(\psi)$, with the generator H given by

$$Hf=\left\{\sum_{\vec{x}\in\mathbb{Z}^d}-\frac{1}{2}\gamma\frac{\partial^2}{\partial\varphi(\vec{x})^2}+\frac{1}{2}\frac{\partial S}{\partial\varphi(\vec{x})}\frac{\partial}{\partial\varphi(\vec{x})}\right\}f, \qquad (3)$$

where H is Hermitian, positive in $L^2(d\mu)$, $d\mu\equiv e^{-S(\varphi)/\gamma}d\varphi/$(normalization factor). $f=1$ is the eigenfunction with zero eigenvalue. In terms of a Schrödinger type operator we have

$$L=UHU^{-1}$$
$$=\sum_{\vec{x}\in\mathbb{Z}^d}\left\{-\frac{1}{2}\gamma\frac{\partial^2}{\partial\varphi(\vec{x})^2}+\frac{1}{4}\left[\frac{1}{2\gamma}\left(\frac{\partial S}{\partial\varphi(\vec{x})}\right)^2-\frac{\partial^2 S}{\partial\varphi(\vec{x})^2}\right]\right\}, \qquad (4)$$

where U is the unitary operator $Uf(\varphi)=Z^{-1/2}\exp[-S/2\gamma]f(\varphi)$ from $L^2[d\mu(\varphi)]$ to $L^2(d\varphi)$ (Z is the normalization). Hence, a Feynman-Kac formula follows [9]:

$$(\Omega,f_1e^{-(t_2-t_1)H}f_2\ldots e^{-(t_n-t_{n-1})H}f_n\Omega)_{L^2(d\mu)}$$
$$=(U\Omega,f_1e^{-(t_2-t_1)L}f_2\ldots e^{-(t_n-t_{n-1})L}f_nU\Omega)_{L^2(d\varphi)}$$
$$=\int f_1[\varphi(t_1)]\ldots f_n[\varphi(t_n)]d\rho, \qquad (5)$$

where $\Omega(\varphi)=1$ is the ground state of H, $f_1,f_2,\ldots$ are functions of φ, $t_1\leqslant t_2\leqslant\ldots\leqslant t_n$, and $d\rho=e^{-W}d\nu/\int e^{-W}d\nu$, with

$$W(\varphi)=\int_{-\infty}^{\infty}dt\sum_{\vec{x}\in\mathbb{Z}^d}\left\{\frac{\lambda}{4\gamma}\mathcal{P}'(\varphi(\vec{x},t))[(-\Delta+m^2)\varphi](\vec{x},t)\right.$$
$$\left.+\frac{\lambda^2}{8\gamma}\mathcal{P}'(\varphi(\vec{x},t))^2-\frac{\lambda}{4}\mathcal{P}''(\varphi(\vec{x},t))\right\}, \qquad (6)$$

where $'$ means the derivative in relation to φ, and $d\nu$ is a Gaussian measure with mean zero and covariance given by

$$\gamma C(\vec{x},t;\vec{y},t')\equiv\frac{\gamma}{(2\pi)^{d+1}}\int_{-\infty}^{\infty}dp_0$$
$$\times\int_{T_d}d^dp\frac{\exp[ip_0(t-t')+i\vec{p}\cdot(\vec{x}-\vec{y})]}{p_0^2+\left(\sum_{i=1}^{d}(1-\cos p_i)+\frac{m^2}{2}\right)^2}, \qquad (7)$$

T_d is the torus $(-\pi,\pi]^d$.

Now we study the two-point function $S(x,y)$ up to second order in the coupling constant λ, where

$$S(x,y)\equiv\langle\varphi(x)\varphi(y)\rangle\equiv\int\varphi(x)\varphi(y)d\rho,$$

$x\equiv(x_0,\vec{x})$, $x_0\equiv t\in\mathbb{R}$, $\vec{x}\in\mathbb{Z}^d$. To make easier the calculations we write the polynomial interaction $\mathcal{P}$ in terms of Wick ordered monomials (ordered with respect to the covariance γC): $\mathcal{P}(\varphi)=\sum_{n=2}^{N}a_{2n}/(2n)!:\varphi^{2n}:$. Thus, straightforward (but tedious) calculations give us $S(x,y)$. To understand the S time decay, we search for the singularities in the Fourier transform $\hat{S}(ik_0,\vec{k}=\vec{0})$ determining the zeroes of $\hat{\Gamma}(ik_0,\vec{k}=\vec{0})$, where $\Gamma(x,y)$ is the inverse convolution of the two-point function $S(x,y)$. The value of k_0 gives us the one-particle mass M (which determines the two-point exponential time relaxation rate). We get

$$M=\frac{m^2}{2}\left\{1-\left(\frac{\lambda}{m^2}\right)^2\sum_{n=2}^{N}\left[\frac{a_{2n}^2}{4n(2n-2)!}\left(\frac{\gamma}{m^2}\right)^{2n-2}\right.\right.$$
$$\left.\left.+\frac{a_{2n}a_{2n+2}}{4n(2n-1)!}\left(\frac{\gamma}{m^2}\right)^{2n-1}\right]\right\}$$
$$\times\left[1+O\left(\frac{d}{m^2}\right)\right]+O\left[\left(\frac{\lambda}{m^2}\right)^3\right], \qquad (8)$$

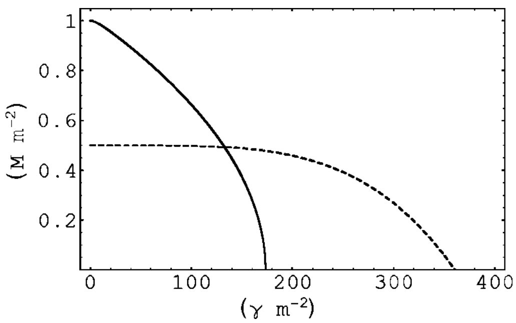

FIG. 1. Curve M (one-particle mass for dashed line, and two-particle bound state mass for full line) vs γ (noise intensity). $m^2/2$ is the one-particle "bare mass." $d=1$, $m=10$, $\lambda=0.001$, $a_4=-1$, $a_6=0.1$, and $\mathcal{P}(\varphi)=a_4/4!:\varphi^4:+a_6/6!:\varphi^6:$.

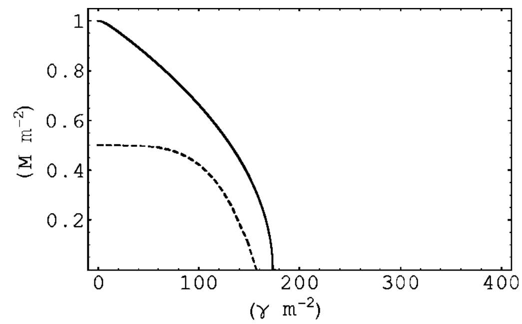

FIG. 2. Curve M (one-particle mass for dashed line, and two-particle bound state mass for full line) vs γ (noise intensity). $m^2/2$ is the one-particle "bare mass." $d=1$, $m=10$, $\lambda=0.001$, $a_4=-1$, $a_6=0.5$, and $\mathcal{P}(\varphi)=a_4/4!:\varphi^4:+a_6/6!:\varphi^6:$.

(assume above $a_{2N+2}=0$). The expression above shows that the one-particle mass also decreases with γ, but, according to Ref. [1] (see the Fig. 1 and Fig. 2 below), the bound state mass decreases more rapidly: e.g., taking as the initial potential $\mathcal{P}(\varphi)=a_4:\varphi^4:+a_6:\varphi^6:$, for small noise and $d=1$, both masses decay as $c_0-c_1\lambda^2$, where the coefficient c_1 depends on γ and m; however, c_1 for the one-particle mass is given by the coefficient c_1 describing the behavior of the bound state mass times a factor $1/m^2$. We remark that it is happening in a region where the perturbative approach is still acceptable. To make it clear, we rescale the fields to write the covariance γC as C in the expression for S above. Thus, we get (also using some well known properties of the Wick ordering [9])

$$S(x,y)=\int \gamma\phi(x)\phi(y)d\tilde{\rho},$$

where $d\tilde{\rho}=e^{-\tilde{W}}d\tilde{\nu}/\int e^{-\tilde{W}}d\tilde{\nu}$, with

$$\begin{aligned}\tilde{W}(\phi)=\int_{-\infty}^{\infty}dt\sum_{\vec{x}\in\mathbb{Z}^d}\sum_{n=2}^{N}\sum_{k=2}^{N}\\ \times\Bigg\{\frac{\lambda\gamma^{(n-1)}}{4}\frac{a_{2n}}{(2n-1)!}:\phi^{(2n-1)}(\vec{x},t)\\ \times[(-\Delta+m^2)\phi](\vec{x},t):+\frac{\lambda^2}{8}\sum_{k=2}^{N}\gamma^{(k+n-2)}\\ \times\frac{a_{2n}a_{2k}}{(2n-1)!(2k-1)!}:\phi^{(2n-1)}(\vec{x},t)::\phi^{(2k-1)}(\vec{x},t):\Bigg\},\end{aligned}$$

and $d\tilde{\nu}$ is now the Gaussian measure with covariance C (instead of γC). Remember that the bound for the massive covariance C involves a factor $1/m^2$ [see Eq. (7)]. If we also rescale the fields using these factors, i.e., if we replace ϕ by ψ/m, we get a potential $W(\psi)$ with the main coefficients involving $\lambda(\gamma/m^2)$. Hence, it is transparent that the expression above involves a small perturbation of the Gaussian measure with covariance C (for properly chosen noise intensity: e.g., $\gamma<m^2$), which supports our results. Note that the one-particle mass dependence on the space dimension d is insignificant, while the dependence of the two-particle bound state mass is drastic: for $d=3$ the existence of bound state is expected only in the extrapolation of large noise [1]. In relation to the sign of the GL potential coefficients, we also note the same fact: the bound state is quite sensitive to the sign of the quartic term, but not the one-particle mass.

In a few words, the scenario described above indicates (at least for $d=1$ and 2) a possible masses crossover and/or same phase transition (since M is going to zero as γ increases).

Now we investigate further spectral structures in the two-particle sector, searching for resonances and antibound states. We take the truncated four-point function

$$\begin{aligned}D(x_1,x_2;x_3,x_4)\equiv\langle\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\rangle\\ -\langle\varphi(x_1)\varphi(x_2)\rangle\langle\varphi(x_3)\varphi(x_4)\rangle.\end{aligned}$$

Due to translation invariance, D depends only on difference variables, and so, we introduce $\xi=x_2-x_1$, $\eta=x_4-x_3$, $\tau=x_3-x_2$. It is useful to study D in terms of the BS equation $D=D_0+DKD_0$, where

$$\begin{aligned}D_0(x_1,x_2;x_3,x_4)\equiv\langle\varphi(x_1)\varphi(x_3)\rangle\langle\varphi(x_2)\varphi(x_4)\rangle\\ +\langle\varphi(x_1)\varphi(x_4)\rangle\langle\varphi(x_2)\varphi(x_3)\rangle,\end{aligned}$$

and the BS kernel $K(x_1,x_2;x_3,x_4)$ is given by the sum of all (channel) two-particle irreducible connected Feynman diagrams with four (amputated) external lines. Taking the Fourier transform, the BS equation gives us

$$\tilde{D}(k)=\tilde{D}(k)[1-(2\pi)^{-2(d+1)}\tilde{K}(k)\tilde{D}_0(k)]^{-1}, \qquad (9)$$

where the notation means $(\tilde{D}_\lambda(k)f)(p)\equiv\int_{-\infty}^{\infty}dq_0\int_{T_d}d^dq\tilde{D}_\lambda(p,q,k)f(q)$, etc., where p,q,k are the conjugate variables of ξ,η,τ. As described in Ref. [1], the singularities of $\tilde{D}(k_0,\vec{k}=\vec{0})$ on the whole complex plane, except on the segments $[m^2,m^2+4d]$ and $[-m^2-4d,-m^2]$ in the imaginary axis, must come from the zeroes of $1-(2\pi)^{-2(d+1)}\tilde{K}(k)\tilde{D}_0(k)$ (here, we analyze only the spectrum for zero momentum $\vec{k}=\vec{0}$). Thus, the spectrum is determined by the values of k_0 which makes equal to 1 the eigenvalues of $(2\pi)^{-2(d+1)}\tilde{K}(k_0)\tilde{D}_0(k_0)$. We have

$$eigenvalues\ = -\tfrac{3}{4}a_4\lambda\gamma(2\pi)^{-d}\mu_{\pm}, \quad \mu_{\pm}=\alpha\pm\sqrt{\beta\delta}, \tag{10}$$

$$\alpha(k_0)\equiv\int_{T_d}d^dq\frac{1}{E_0(\vec{q})^2+(k_0^2/4)},$$

$$\beta(k_0)\equiv\int_{T_d}d^dq\frac{1}{E_0(\vec{q})[E_0(\vec{q})^2+(k_0^2/4)]},$$

$$\delta(k_0)\equiv\int_{T_d}d^dq\frac{E_0(\vec{q})}{E_0(\vec{q})^2+(k_0^2/4)}$$

$$eingenvectors\ = \sqrt{\delta}\pm\sqrt{\beta}E_0(\cdot),$$

$$E_0(\vec{p})=\sum_{j=1}^{d}(1-\cos p_j)+m^2/2.$$

Hence, considering the action of $\tilde{D}(k_0)$ on the space generated by the eigenvectors we get

$$\{\sqrt{\delta}\pm\sqrt{\beta}E_0(\cdot),\tilde{D}_\lambda(k_0)[\sqrt{\delta}\pm\sqrt{\beta}E_0(\cdot)]\} = \frac{\pm(2\pi)^{d+2}\gamma^2\sqrt{\beta\delta}\mu_{\pm}}{1+\frac{3}{4}(2\pi)^{-d}a_4\lambda\gamma\mu_{\pm}}, \tag{11}$$

which, together with the analyticity properties of α, β, δ, etc., gives us the equation for the bound state masses (singularities) in terms of the noise strength γ

$$\mu_{\pm}(k_0)=-4(2\pi)^d/3a_4\lambda\gamma, \tag{12}$$

(i.e., *eigenvalues* $=1$). As previously referred to, a detailed analysis of these masses is presented in Ref. [1]. Here, to search for further points in the spectrum determining, e.g., resonances and antibound states, we consider the analytic continuation of $\mu_{\pm}(k_0)$ onto a second (Riemann) sheet in k_0, and study the possible singularities there. We will be restricted to the case $d=1$, where the expression for $\mu_{\pm}$ may be explicitly computed. For $d=1$, we have

$$\mu_{\pm}(z)=(2\pi i/z)[B(z)-B(-z)]\pm(2\pi i/z)[B(z)+B(-z)] \times\left[1-\frac{2B(0)}{B(z)+B(-z)}\right]^{1/2}, \tag{13}$$

where

$$B(z)=\{(1+\tfrac{1}{2}[m^2+iz])^2-1\}^{-1/2}.$$

Here, we are interested only in the singularities close to $2M\approx m^2$ (where M is the one particle mass; this is enough to describe resonances and antibound states with directly physical interpretation). Writing $z=im^2+\varepsilon$, we obtain

$$\mu_+=\frac{4\pi}{m^2}\left[\left(1+\frac{i\varepsilon}{2}\right)^2-1\right]^{-1/2}+\mathcal{O}(1/m^4).$$

The function above, specifically: $F(\cdot)=[(\cdot)^2-1]^{1/2}$, has a continuation to a two-sheeted manifold through the cut $z\in[m^2,m^2+4]$ (where $z=im^2+\varepsilon$). Thus, $\mu_{\pm}(z)=-4(2\pi)^d/3a_4\lambda\gamma$, for $a_4<0$ has one solution with z in the imaginary axis below m^2 in the first sheet (it gives the two-particle bound state described in Ref. [1]), and no solution in the second sheet close to m^2, which implies the absence of resonances and antibound states. For $a_4>0$, there is no bound state (for small λ and γ not large) but there is one antibound state (solution in the imaginary axis "below m^2," in the second sheet).

Turning to $\mu_-(z)$, for $z=im^2+\varepsilon$ ($|\varepsilon|$ small), the expression is analytic and so limited, which leads to the absence of solutions for $\mu_-(z)=-4(2\pi)^d/3a_4\lambda\gamma$ if we take λ small enough (and γ not large).

In conclusion, we have seen (at least for $d=1$ and 2) that increasing the noise strength the difference between the one-particle and the bound state masses certainly becomes smaller, which indicates the possibility of a crossover for larger noise and/or a phase transition, since the one-particle mass M is going to zero). We remark that the perturbative analysis for noise not very large ($\gamma\ll m^2$) is supported by rigorous results [6]. For $d=1$, in the ladder approximation, we also have shown that there are no resonances but there is one antibound state close to the two particle mass threshold for a positive quartic term in the GL potential, which indicates the possibility of a bound state due to changes in the interactions (changes caused, e.g., by increasing the noise intensity).

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